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PROJECT ADMINISTRATION DATA SHEET

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Project Director: K. R. Davey School/~~Lab~~ Elect. Engr.

Sponsor: Naval Coastal Systems Center, Panama City, FL 32407

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ADMINISTRATIVE DATA

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COMMENTS:

*E-21-G05 replaces Advance Project Number E-21-E55 which has been cancelled.

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SPONSORED PROJECT TERMINATION SHEET

Date May 18, 1983

Project Title: T-Omega Method Integral Technique and Its Application in Magnetoquasistatic and Acoustical Problems

Project No: E-21-G05

Project Director: K. R. Davey

Sponsor: Naval Coastal Systems Center

Effective Termination Date: 7/1/83

Clearance of Accounting Charges: 7/1/83

Grant/Contract Closeout Actions Remaining:

- [x] Final Invoice ~~and Closing Documents~~
- [ ] Final Fiscal Report
- [x] Final Report of Inventions
- [x] Govt. Property Inventory & Related Certificate
- [x] Classified Material Certificate
- [ ] Other ______________________

Assigned to: Elect. Engr. (School/~~Laboratory~~)

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FINAL REPORT

ACOUSTIC WAVE SCATTERING USING A NEW GUAGE TRANSFORMATION – THE S–Ω TECHNIQUE

By
Kent R. Davey

March 15, 1983

GEORGIA INSTITUTE OF TECHNOLOGY
A UNIT OF THE UNIVERSITY SYSTEM OF GEORGIA
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ACOUSTIC WAVE SCATTERING USING A NEW GAUGE TRANSFORMATION - THE S-Ω TECHNIQUE

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March 15, 1983

ABSTRACT

Formulation of the acoustic scattering problem is considered with the aim of addressing the prolate spheroidal scatterer. Prolate spheroidal coordinates, operations, and basis functions are reviewed. A new formulation of the basic equations is developed expressing elastic body displacements as the sum of another vector and the gradient of a scalar ($\overline{u}_e = \overline{s} = -\nabla\Omega$). Scattering from a spheroidal body is developed and analyzed with this new formulation. The formulation's usefulness in terms of reducing the acoustic wave equation to one vector and one scalar Helmholtz equation is employed in addressing the scattering from a prolate spheroidal body. Although point matching is required, all surface integrals are reduced by orthogonality conditions. An alternate formulation independent of the $\overline{s}$-Ω technique is introduced for comparison and seen to involve more complex surface integrals but fewer unknowns than the s-Ω technique.

I. INTRODUCTION

The problem at hand is the solution for acoustical wave scattering off an elastic body immersed in a fluid. The matrix theory for scattering of acoustic [1] and electromagnetic [5] waves by single obstacles of arbitrary shape fostered by Waterman has been extended to the elastic case by Waterman [3] and Varathavajulu and Pao [4]. The extension of this work into the problem of an elastic obstacle in a fluid [2] has met with limited success. The method which has come to be known as the T-matrix technique involves global expansions using Legendre polynomials and spherical Hankel functions; the technique works quite well for nearly spherical objects.

It is with the motivation of solving the scattering from a prolate spheroid that the present technique was developed, although application of the basic formulation is not limited to such cases. The intent was to break down the elastic equation into a simpler representation, one that is not so limited in its separability. The new representation it will be seen is separable in thirteen coordinate systems. This is accomplished at the cost of an additional gauge constraint and one additional freedom because of redundancy introduced.

II. BASIC EQUATIONS

For the problem at hand, we consider an infinite homogeneous fluid of density ρ and Lame' parameter λ (where sound velocity $c^2 = \lambda/\rho$). The elastic body with density ρ_o and Lame' parameters λ_o and μ_o is embedded in the fluid. A monochromatic wave with frequency ω, displacement $\overline{u}_i$ is incident upon the obstacle; a scattering wave $\overline{u}_s$ results. The total field $\overline{u} = \overline{u}_i + \overline{u}_s$ is more conveniently expressed as the negative gradient of a scalar $-\nabla\Omega_f$. The defining equation for Ω becomes

$$\nabla^2\Omega_f + k^2\Omega_f = 0 \tag{1}$$

where $k^2 = \rho\omega^2/\lambda$. The elastic body wave equation is

$$\frac{\nabla(\nabla \cdot \overline{u})}{k_o^2} - \frac{\nabla \times \nabla \times \overline{u}}{k_1^2} + \overline{u} = 0 \tag{2}$$

where

$$k_o^2 = \rho_o\omega^2/(\lambda_o + 2\mu_o)$$

$$k_1^2 = \rho_o\omega^2/\mu_o$$

Because the fluid is assumed inviscid, the relevant boundary conditions between the fluid and elastic body are

$$\hat{n} \cdot \|\overline{u}\| = 0 \tag{3}$$

$$\hat{n} \cdot \|\overline{t}\| = 0 \tag{4}$$

$$\hat{n} \times \overline{t}_{elastic} = 0 \tag{5}$$

where $\hat{n}$ denotes the outward interfacial normal, the double bars $\| \;\|$ denote jump in the quantity within across the interface, and $\overline{t}_{f(e)}$ is the surface traction in the fluid (elastic) body. The traction is derived from

$$\overline{t}_f = \hat{n} \cdot \overline{\overline{T}}_f = \hat{n} \cdot (\lambda\overline{\overline{I}}\nabla \cdot \overline{u}) \tag{6}$$

$$\bar{t}_e = \hat{n} \cdot \bar{\bar{T}}_e = \hat{n} \cdot (\lambda_o \bar{\bar{I}} \nabla \cdot \bar{u} + \mu_o(\nabla\bar{u} + \bar{u}\nabla)) \tag{7}$$

III. THEORETICAL FORMULATION $\bar{u} = \bar{s} - \nabla\Omega$

With the aim of simplifying (2), we adopt a rather non-standard representation of the elastic displacement $\bar{u}_e$ as the sum of another vector $\bar{s}$ and the gradient of a scalar.

$$\bar{u}_e = \bar{s} - \nabla\Omega_e \tag{8}$$

Substitution of (8) into (2) yields the new defining equation

$$\nabla^2\bar{s} + k_1^2\bar{s} = 0 \tag{9}$$

with the gauge

$$\nabla^2\Omega_e + k_o^2\Omega_e = \left(\frac{k_1^2 - k_o^2}{k_1^2}\right) \nabla \cdot \bar{s} \tag{10}$$

Observe that we have reduced the elastic wave equation to one vector Helmholtz equation and one inhomogeneous scalar equation; the homogeneous part of these equations involves only one wavenumber each. In the special case where $k_1 = k_o$, the scalar equation becomes homogeneous as well. The advantage this technique has over the standard representation of $\bar{u}_e$ as $(\nabla \times \bar{A} - \nabla\Omega)$ develops in the application of the boundary conditions. In spherical scattering problems, the basis functions are orthogonal over the obstacle surface, and this orthogonality does not depend on the wavenumbers k. In more complex systems, the orthogonality of the basis set, if it exists, does depend on the wavenumbers; point matching to satisfy boundary conditions is not possible. An integral expression of the elastic and fluid unknowns becomes a most convenient tool in these cases. Redundancy in the formulation of $\bar{u}_e$ allows an arbitrary condition to be placed on the parameter $\bar{s}$ or Ω in the volume (such as $\nabla \cdot \bar{s} = 0$, $\frac{\partial \Omega_e}{\partial n} = 0$, or $\Omega_e = \Omega_f$). Appropriate use of this function greatly reduces the difficulty of matching boundary conditions. Application of this formalism to both a spherical and spheroidal elastic body are discussed below.

IV. APPLICATION OF THE FOMRALISM TO THE SPHERICAL PROBLEM

Before examining the details of the more difficult spheroidal scatterer, consider the extension of the formalism to the solid spherical body scatterer. We begin by expanding Ω_f, Ω_e, and $\bar{s}$ in terms of the appropriate basis functions, i.e.,

$$\Omega_f = \sum \{a_{nm\sigma} J_n(kr) Y_{nm\sigma}(\theta,\phi) + b_{nm\sigma} H_n(kr) Y_{nm\sigma}(\theta,\phi)\} \tag{11}$$

$$\Omega_e = \sum \{C_{nm\sigma} J_n(k_o r) Y_{nm\sigma}(\theta,\phi) + d_{nm\sigma} J_n(k_1 r) Y_{nm\sigma}(\theta,\phi)\} \tag{12}$$

$$\bar{s} = \sum_{\tau=1}^{3} \sum \{g_{\tau nm\sigma} \operatorname{Re}[\Psi_{\tau nm\sigma}]\} \tag{13}$$

where

J_n = spherical Bessel function of order n

H_n = spherical Hankel function of order n.

$$Y_{nm\sigma}(\theta,\phi) = \left[\varepsilon_m\left(\frac{2n+1}{4\pi}\ \frac{(n-m)!}{(n+m)!}\right)\right]^{1/2} P_n^m(\cos\theta)\begin{pmatrix}\cos m\phi\\ \sin n m\phi\end{pmatrix}\ ;\ \sigma = e,0$$

ε_m = Newmann factor

σ = Azimuthal parity (even or odd)

m = 0,1,...,n specifies rank of the spherical harmonic

n = 0,1,...,n specifies order of the spherical harmonic.

$$\overline{\Psi}_{1nm\sigma} = \nabla\left[H_n(k_1r)Y_{nm\sigma}(\theta,\phi)\right]$$

$$\overline{\Psi}_{2nm\sigma} = \frac{1}{k_1}\ \nabla x\left\{\overline{r}H_n(k_1r)Y_{nm\sigma}(\theta,\phi)\right\}$$

$$\overline{\Psi}_{3nm\sigma} = \frac{1}{k_1}\ \left(\nabla\ x\ \overline{\Psi}_{2nm\sigma}\right)$$

$a_{nm\sigma}$ = appropriate expansion constants for the incident wave.

$b_{nm\sigma}$ = expansion constants for the outgoing wave.

Because the Legendre polynomials are orthogonal over a spherical surface, the unknowns can be determined quite easily. Note first that the second term in the expansion for Ω_e is needed to satisfy the inhomogeneous part of (10); thus, $d_{nm\sigma}$ is dependent on $g_{\tau nm\sigma}$ through the gauge. There are essentially 5 sets of unknowns - b, c, and g_τ. The boundary conditions (3), (4) and (5) yield the 4 requirements

$$s_r - \frac{\partial\Omega_e}{\partial r} = -\ \frac{\partial\Omega_f}{\partial r} \tag{14}$$

$$\lambda(\nabla\cdot\overline{s} - \nabla^2\Omega_e) + 2\mu\left(\frac{\partial s_r}{\partial r} - \frac{\partial^2\Omega_e}{\partial r^2}\right) = -\lambda k^2\Omega_f \tag{15}$$

$$\frac{\partial s_\theta}{\partial r} - \frac{1}{r}\ \frac{\partial s_r}{\partial\theta} - \frac{\partial}{\partial r}\left(\frac{1}{r}\ \frac{\partial\Omega_e}{\partial\theta}\right) + \frac{1}{r}\ \frac{\partial^2\Omega_e}{\partial\theta\partial r} = 0 \tag{16}$$

$$\frac{\partial S_\phi}{\partial r} - \frac{\partial}{\partial r}\left(\frac{1}{r\sin\theta}\frac{\partial \Omega_e}{\partial \phi}\right) - \frac{1}{r\sin\theta}\frac{\partial S_r}{\partial \phi} + \frac{1}{r\sin\theta}\frac{\partial^2 \Omega_e}{\partial r \partial \phi} = 0 \tag{17}$$

There is but one additional condition necessary to realize a complete solution. Two convenient choices appear to be:: (a) $\frac{\partial \Omega_e}{\partial n} = \left.\frac{\partial \Omega_f}{\partial n}\right|_S$, and (b) $\nabla \cdot S = 0$ on the surface.

Independent of choice, solution proceeds by substituting the expansions for Ω_f, Ω_e, and $\overline{S}$ into these 6 conditions (including gauge), multiplying both sides of $Y_{n'm'\sigma'}$ and integrating over the surface. This essentially amounts to solving a 6 x 6 matrix for each indice set since

$$\int\int Y_{nm\sigma}\, Y_{n'm'\sigma'}\, ds = \delta_{nn'}\delta_{mm'}\delta_{\sigma\sigma'} \tag{18}$$

V. SPHERICAL OBSTACLE SCATTERING

The most serious problem faced when leaving the simpler geometric objects is that the orthogonality of the coordinate basis sets, if it exists at all, is dependent on the wave number of the medium. The spheroidal wave function satisfying the scalar wave equation is one such example where the basis set is orthogonal and dependent on k; solutions to the vector Helmholtz equation are however, not orthogonal. By way of illustrating the technique's versatility, we now work out the scattering problem of the elastic spheroid immersed in a fluid.

A. Prolate Spheroidal Coordinates

With the intent of accurately partraying the problem and thus giving a better understanding of the motivation behind the solution suggested, a brief review of the prolate spheroidal coordinate system is in order. The spheroidal coordinate system (u,v,ϕ) is related to the Cartesian system (x,y,z) as follows:

$$x = \ell\{(1 - v^2)(u^2 - 1)\}^{1/2} \cos\phi \tag{19}$$

$$y = \ell\{(1 - v^2)(u^2 - 1)\}^{1/2} \sin\phi \tag{20}$$

$$z = \ell\, u\, v \tag{21}$$

where ℓ is the semifocal length. A representation of these coordinates is shown in Fig. 1. Note that in the prolate system

$$1 \leq u < \infty \text{ and } -1 \leq v \leq 1$$

The coordinate u is the radial coordinate, v the angular coordinate, and ϕ the azimuthal coordinate as in cylindrical or spherical systems. As $u \to \infty$ the constant surfaces u become spherical and it can be shown that

$$\ell u \to r$$

$$v \to \cos\theta$$

where r and θ are the usual spherical coordinates.

An element of length is defined as

$$(ds)^2 = (dx)^2 + (dy)^2 + (dz)^2 = h_u^2\, du^2 + h_v^2 dv^2 + h_\phi^2 d\phi^2$$

where

$$h_u = \ell \left(\frac{u^2 - v}{u^2 - 1}\right)^{1/2}$$

$$h_v = \ell \left(\frac{u^2 - v^2}{1 - v^2}\right)^{1/2}$$

$$h_\phi = \ell \left\{(1 - v^2)(u^2 - 1)\right\}^{1/2}$$

using these relationships, one can derive the gradient, divergence, curl, and Laplacian necessary in the forthcoming development.

$$\nabla_u \psi = \frac{1}{h_u} \frac{\partial \psi}{\partial u}, \quad \nabla_u \psi = \frac{1}{h_v} \frac{\partial \psi}{\partial v}, \quad \nabla_\phi \psi = \frac{1}{h_\phi} \frac{\partial \psi}{\partial \phi} \tag{22}$$

$$\nabla \cdot \overline{A} = \frac{1}{h_u h_v h_\phi} \left\{\frac{\partial}{\partial v} (h_v h_\phi A_u) + \frac{\partial}{\partial v} (h_u h_\phi A_v) + \frac{\partial}{\partial \phi} (h_u h_v A_\phi)\right\} \tag{23}$$

$$(\nabla \times \overline{A})_u = \frac{1}{h_v h_\phi} \left\{\frac{\partial}{\partial v} (h_\phi A_\phi) - \frac{\partial}{\partial \phi} (h_v A_v)\right\} \tag{24}$$

$$\nabla^2 \psi = \frac{1}{h_u h_v h_\phi} \left\{\frac{\partial}{\partial u} \left(\frac{h_v h_\phi}{h_u} \frac{\partial \psi}{\partial u}\right) + \frac{\partial}{\partial v} \left(\frac{h_\phi h_u}{h_v} \frac{\partial \psi}{\partial v}\right) + \frac{\partial}{\partial \phi} \left(\frac{h_u h_v}{h_\phi} \frac{\partial \psi}{\partial \phi}\right)\right\} \tag{25}$$

B. Prolate Sperhoidal Wave Functions

In spheroidal coordinates the scalar wave equation

$$\nabla^2 \psi + k^2 \psi = 0 \tag{26}$$

becomes

$$\left\{\frac{\partial}{\partial u} (u^2 - 1) \frac{\partial}{\partial u} + \frac{\partial}{\partial v} (1 - v^2) \frac{\partial}{\partial v} + \frac{(u^2 - v^2)}{(u^2 - 1)(1 - v^2)} \frac{\partial^2}{\partial \phi^2} + q^2 (u^2 - v^2)\right\}\psi = 0 \tag{27}$$

where $q = k\ell$. The basic solutions of (27) are

$$\psi_{m,n} = S_{m,n}(q,v) R_{m,n}(q,v) \begin{matrix}\cos\\ \sin\end{matrix}(m\phi) \tag{28}$$

where $R_{m,n}$ and $S_{m,n}$ satisfy

$$\frac{d}{du}\left\{(u^2 - 1)\frac{d}{du} R_{m,n}\right\} - \left\{\lambda_{m,n} - q^2u^2 + \frac{m^2}{u^2 - 1}\right\} R_{m,n} = 0 \tag{29}$$

and

$$\frac{d}{dv}\left\{(1 - v^2)\frac{d}{dv} S_{m,n}\right\} + \left\{\lambda_{m,n} - 2^2v^2 - \frac{m^2}{1 - v^2}\right\} S_{m,n} = 0 \tag{30}$$

m,n are integers.

$\lambda_{m,n}$ are the eigenvalue separation constants admitting solutions at $v = \pm 1$.

According to Flammer (1957), appropriate expansions for $R_{m,n}$ and $S_{m,n}$ are as follows:

$$R^{\substack{(1)\\(2)}}_{m,n}(q,u) = \frac{1}{\sum\limits_{r=0,1,\ldots}^{x} d_r^{m,n}(q)\,\frac{(2m+r)!}{r!}} \left(\frac{u^2 - 1}{u^2}\right)^{m/2}$$

$$\sum_{r=0,1\ldots}^{x} i^{r+m-n} d_r^{m,n}(q)\,\frac{(2m+r)!}{r!} \left\{\begin{matrix} j_{m+r}(qu)\\ y_{m+r}(qu)\end{matrix}\right. , \tag{31}$$

where

$$j_n(z) = \left(\frac{\pi}{2z}\right)^{1/2} J_{n+1/2}(z) ,$$

and

$$y_n(z) = \left(\frac{\pi}{2z}\right)^{1/2} Y_{n+1/2}(z)$$

are related to the conventional cylindrical Bessel functions as indicated.

The radial functions of the third and fourth type are obtained from

$$R^{(3)}_{m,n} = R^{(1)}_{m,n} + iR^{(2)}_{m,n} \tag{32}$$

and

$$R^{(4)}_{m,n} = R^{(1)}_{m,n} - iR^{(2)}_{m,n} \tag{33}$$

$$S_{m,n}(q,v) = \sum_{r=0,1,2,..}^{\infty}{}' \; d_r^{m,n}(q) P^m_{m+r}(v) \; . \tag{34}$$

The prime over the summation indicates that the summation is over (only) even values when n-m is even and over (only) odd values when n-m is odd. The associated Legendre functions employed here are defined by

$$P^m_n(v) = (1 - v^2)^{m/2} \frac{d^m P_n(v)}{dv^m} , \tag{35}$$

for $-1 \leq v \leq 1$ where $P_n(v)$ is a Legendre polynomial of order n. The angle functions are normalized such that

$$S_{m,n}(q,0) = P^m_n(0) , \tag{36}$$

and, furthermore,

$$\lambda_{m,n}(0) = n(n + 1) , \quad n \geq m . \tag{37}$$

The parameter $d_r^{m,n}(q)$ is defined by the iterative formula

$$\frac{(2m + r + 2)(2m + r + 1)q^2}{(2m + 2r + 3)(2m + 2r + 5)} d^{mn}_{r+2}(q)$$

$$+ \left\{ (m+r)(m+r+1) - \lambda_{mn}(q) + \frac{\left(2(m+r)(m+r+1) - 2m^2 - 1\right)q^2}{(2m + 2r - 1)(2m + 2r + 3)} \right\} d^{mn}_r(q)$$

$$+ \frac{r(r-1)q^2}{(2m+2r-3)(2m+2r-1)} d^{mn}_{r-2}(q) = 0 \quad (r > 0) \tag{38}$$

and is also tabulated by Fammer (1957).

From the general theory of Sturm-Liouville differential equations, it follows from (30) that the functions $S_{m,n}(q,v)$ form an orthogonal set on the interval $-1 < v < +1$. Thus,

$$\int_{-1}^{+1} S_{m,n}(q,v) S_{m,n'}(q,v)\,dv = \begin{cases} N_{m,n} & \text{if } n = n' \\ 0 & \text{if } n \neq n' \end{cases} \tag{39}$$

where $N_{m,n}$ is easily found with the use of the normalization factor of the associated Legendre function. To be explicit,

$$N_{m,n} = 2 \sum_{r=0,1,2,\ldots}^{\infty} \frac{(r+2m)!\,(d_r^{m,n})^2}{(2r+2m+1)\,r!} \tag{40}$$

(In what follows, we shall not show the prime in summations over r. Thus, in these cases, we can regard the summation to extend over all values of r but $d_r^{m,n}$ is zero for even (odd) values of r when n-m is odd (even).)

The Green's function of the scalar wave equation

$$\nabla^2 + k^2) G(\bar{r},\bar{r}') = -\delta(\bar{r} - \bar{r}') \tag{41}$$

is

$$\frac{e^{ik|\bar{r}-\bar{r}'|}}{4\pi|\bar{r}-\bar{r}'|} = \frac{ik}{2\pi} \sum_{m=0}^{\infty} \sum_{n=m}^{\infty} \frac{(2-\delta_{om})}{N_{mn}} S_{mn}(q,v) S_{mn}(q,v') \cos m(\phi-\phi)$$

$$\begin{cases} R^{(1)}_{mn}(q,u)\; R^{(3)}_{mn}(q,u'), & u < u' \\ R^{(1)}_{mn}(q,u')\; R^{(3)}_{mn}(q,u), & u > u' \end{cases} \tag{42}$$

where

$$\delta(\overline{r}-\overline{r}') = \frac{\delta(u-u')\delta(v-v')\delta(\phi-\phi')}{h_u h_v h_\phi}$$

Finally, solutions to the vector wave equation

$$(\nabla^2 + k^2)\overline{A} = 0 \tag{43}$$

are written in terms of the scalar wave equations solutions as follows:

$$\overline{A} = \overline{A}_1 + \overline{A}_2 + \overline{A}_3 \tag{44}$$

$$\overline{A}_1 = \nabla\psi \tag{45}$$

$$\overline{A}_2 = \nabla \times \overline{r}\psi \tag{46}$$

$$\overline{A}_3 = \nabla \times \overline{A}_2 \tag{47}$$

where $\overline{r} = x\hat{a}_x + y\hat{a}_y + z\hat{a}_z$. It should be emphasized that the vector wave solutions in any one set are <u>not orthogonal</u> nor are they orthogonal with any other set.

C. A Solution Using the $\overline{S}$-Ω Method

After considerable reflection, it appears that there is sufficient reason to warrant avoiding all integral equations involving the vector portion of the wave solution. We express $\overline{S}$ as an expansion of prolate spheroidal basis functions involving $q_1 = k_1\ell$.

$$\overline{S} = \sum c_{\tau mn\sigma} \operatorname{Re}(\overline{A}_{\tau mn\sigma}) \tag{48}$$

$\overline{A}_{1mn\sigma} = \nabla\psi_{m,n}(q_1)$

$\overline{A}_{2mn\sigma} = \nabla \times \overline{r}\xi_{mn}(q_1)$

$\overline{A}_{3mn\sigma} = \nabla \times \overline{A}_{2mn\sigma}$

σ = even (odd) parity denoting the use of $\cos(m\phi)$ $(\sin(m\phi))$

τ = index summed from 1 to 3

$\psi_{m,n}$ = scalar wave function of (28), using only $R^{(1)}_{m,n}$.

The fluid field is the sum of the incident field expressed in terms of known coefficients $a_{mn\sigma}$ and the scattered field $b_{mn\sigma}$,

$$\Omega_f = \sum_1 a_{mn\sigma}\psi^{(1)}_{m,n}(q) + b_{mn\sigma}\psi^{(3)}_{m,n}(q) \tag{49}$$

where

$q = k\ell$, k = wavenumber of the fluid

$\psi^{(1)}_{mn}(q)$ involves $R^{(1)}_{m,n}$ (Bessel function)

$\psi^{(3)}_{mn}(q)$ involves $R^{(3)}_{m,n}$ (Hankel function).

We eliminate the redundancy by requiring the divergence of $\overline{s}$ to be zero, $\nabla \cdot \overline{s} = 0$, and expand Ω_e and $\frac{\partial\Omega_e}{\partial n}$ on the spheroid surface in terms of $q_o = k_o\ell$ and position $u = u_s$ of the spheroid surface as follows:

$$\Omega_e = \sum d_{m,n,\sigma} S_{mn}(q_o,v) \begin{Bmatrix}\cos m\phi \\ \sin m\phi\end{Bmatrix} (u_s^2 - v^2)^{1/2} \tag{50}$$

$$\frac{\partial\Omega_e}{\partial n} = \sum f_{m,n,\sigma} S_{mn}(q_o,v) \begin{Bmatrix}\cos m\phi \\ \sin m\phi\end{Bmatrix} (u_s^2 - v^2)^{1/2} \tag{51}$$

Solution proceeds by expressing Ω_e in terms of its integral equation

$$\Omega_e(\overline{r}) = \iint (\nabla \underset{0}{\underset{\uparrow}{\cdot}} \overline{s})\left(\frac{k_1^2 - k_o^2}{k_1^2}\right) G\,dv' + \iint \left(\frac{\partial\Omega_e}{\partial n} G - \Omega_e \frac{\partial G}{\partial n}\right) da' \tag{52}$$

where $G = \dfrac{e^{-k_o|\overline{r} - \overline{r}'|}}{4\pi|\overline{r} - \overline{r}'|}$.

Point matching required in realizing boundary conditions is made possible by use of the integral expressions which acts to "smooth out" the basis function expansions on the surface.

The interfacial conditions required for solution are as follows:

$$\nabla \cdot \overline{s} = 0 \tag{53}$$

$$s_n - \frac{1}{h_u}\left(\frac{\partial \Omega_e}{\partial n}\right) = -\frac{1}{h_u}\left(\frac{\partial \Omega_f}{\partial n}\right) \tag{54}$$

$$\lambda_o(\nabla \cdot \overline{s} - \nabla^2 \Omega_e) + 2\mu_o(\nabla_u s_u - \nabla_u(\nabla_u \Omega_e)) = -\lambda \nabla^2 \Omega_f \tag{55}$$

$$\hat{n} \times \left[\hat{n} \cdot (\nabla(\overline{s} - \nabla\Omega_e) + (\overline{s} - \nabla\Omega_e)\nabla)\right] = 0 \tag{56}$$

The first condition (53) gives m x n requirements on $c_{\tau mn\sigma}$ in terms of m x n surface points. Continuity of $\overline{u}$, (54), yields a connection between $c_{\tau mn\sigma}$, $f_{m,n,\sigma}$, $a_{m,n,\sigma}$, and $b_{m,n,\sigma}$. Condition (55) can be simplified to

$$\lambda_o(-k_o^2\Omega_e) + 2\mu_o\left[\frac{1}{h_u}\frac{\partial s_u}{\partial u} - k_o^2\Omega_e + \left(\frac{2u}{u^2 - v^2} - \frac{u(v^2 - 1)}{\sqrt{(u^2 - v^2)(u^2 - 1)^3}}\right)\frac{\partial \Omega_e}{\partial u}\right]$$

$$\frac{1}{h_u h_v h_\phi}\left[\frac{\partial}{\partial v}\left(\frac{h_\phi h_u}{h_v}\frac{\partial}{\partial v}\Omega_e\right) + \frac{\partial}{\partial \phi}\left(\frac{h_u h_v}{h_\phi}\frac{\partial}{\partial \phi}\Omega_e\right)\right] = -\lambda k^2 \Omega_f \tag{57}$$

All of the quantities in (57) can be expressed in terms of the unknowns $c_{\tau mn\sigma}$, $b_{m,n,\sigma}$, $d_{m,n,\sigma}$, and $f_{m,n,\sigma}$. The boundary condition expressed in (56) gives a relation between $c_{\tau,m,n,\sigma}$, $d_{m,n,\sigma}$ and $f_{m,n,\sigma}$. Thus we have 5 equations and 6 unknowns for every point, i.e., c_1, c_2, c_3, b, d, and f. The final condition is the integral relation which accomplishes the smoothing discussed above. Because of the expansion choice in (50) and (51) (with the $(u^2 - v^2)^{1/2}$ term), surface integral in (52) falls out immediately, i.e.,

$$\int_0^{2\pi}\int_{-1}^{1} \left(\frac{\partial \Omega_e}{\partial n} G - \Omega_e \frac{\partial G}{\partial n}\right) h_v dv\, h_\phi d\phi = \frac{ik}{2} \frac{(2 - \delta_{om})}{N_{mn}(q_o)} S_{mn}(q,v) R_{mn}^{(1)}(q,u) \qquad (58)$$

Equation (52) then follows without explicitly evaluating any integrals; this is the final condition necessary for solution. An inversion of the 6 x n x m matrix yields the solution for $b_{m,n,\sigma}$ (the primary parameter of interest). Realistically, m is usually fixed by the incident wave to be 1 so the final matrix becomes 6 x n in size.

VI. ALTERNATE SOLUTION OF THE PROLATE SPHEROIDAL SCATTERER INDEPENDENT OF THE $\overline{S}$ - Ω FORMULATION

A solution of the spheriodal scatterer which develops rather straightforwardly independent of the $\overline{s}$ - Ω method by appropriate expansion of key surface quantities is worth consideration. Suppose we begin by expanding the fluid wave function Ω_f as before,

$$\Omega_f = \sum a_{mn\sigma} \psi^{(1)}_{mn\sigma}(q) + b_{mn\sigma}\psi^{(3)}_{mn\sigma}(q) \qquad (59)$$

where "a" represents the incident wave and "b" the scattered wave. Again, the superscripts (1) and (3) refer to radial "u" dependence involving Bessel and Hankel functions, respectively. Both of these scalar wave functions are normalized so as to be equal to $S_{m,n}(q,v) \begin{cases}\cos m\phi \\ \sin m\phi\end{cases}$ at the spheroidal surface. In addition, we express $\frac{\partial\Omega_f}{\partial n} \equiv \frac{\partial\Omega_f}{\partial u}$ in terms of spheroidal wave functions

$$\left(\frac{1}{h_u}\right) \frac{\partial\Omega_f}{\partial n} = \sum d_{mn\sigma}\, S_{m,n}(q,v) \begin{cases}\cos m\phi \\ \sin m\phi\end{cases} \tag{60}$$

Note that $d_{m,n,\sigma}$ is not independent of "a" and "b"; they are easily related using the orthogonality condition, i.e., multiplying both sides of (60) by $S_{m',n'} \begin{cases}\cos m'\phi \\ \sin m'\phi\end{cases}$ and integrating over v and ϕ.

$$d_{mn\sigma} = \frac{1}{\pi N_{m,n}} \int_0^{2\pi} \int_{-1}^{1} \frac{1}{h_u} \frac{\partial\Omega_f}{\partial n} S_{m,n} \begin{Bmatrix}\cos m\phi \\ \sin m\phi\end{Bmatrix} dv\,d\phi \tag{61}$$

We proceed by expanding both the surface traction $\overline{t}_e$ and displacement $\overline{u}_e$ appropriately. Boundary conditions on normal displacement, normal traction, and tangential traction imply

$$\hat{n} \cdot \overline{u}_e = -\sum d_{mn\sigma}\, S_{m,n}(q,v) \begin{cases}\cos m\phi \\ \sin m\phi\end{cases} \tag{62}$$

$$\hat{n} \cdot \overline{t}_e = k^2 \sum (a_{m,n,\sigma} + b_{m,n,\sigma}) S_{m,n}(q,v) \begin{cases}\cos m\phi \\ \sin m\phi\end{cases} \tag{63}$$

$$n \times \overline{t}_e = 0 \tag{64}$$

The tangential component of $\overline{u}_e$ (u_v, u_ϕ) are expressed as basis expansions in the elastic body wavenumber $k_1 = q/\ell$,

$$u_v = \sum g_{m,n,\sigma}\, S_{m,n}(q_1,v) \begin{cases}\cos m\phi \\ \sin m\phi\end{cases} \tag{65}$$

$$u_\phi = \sum f_{m,n,\sigma} \, S_{m,n}(q_1,v) \begin{Bmatrix} \cos m\phi \\ \sin m\phi \end{Bmatrix} \tag{66}$$

Thus for each m,n index, there are 3 unknowns "b", "g", and "f".

As shown by Pao and Varatharajulu (1976), the elastic body displacement can be determined as the surface integral

$$\iint \overline{t}(x') \cdot \overleftrightarrow{G}(x|x') - \overline{u}(x') \cdot \left[\hat{n}' \cdot \overset{\perp}{\sum}(x|x')\right]ds' = \begin{cases} \overline{u}_e(x), & x \text{ inside } s \\ 0, & x \text{ not in } s \end{cases} \tag{67}$$

where

$$\overleftrightarrow{G}_{mn}(x|x') = \frac{1}{4\pi\ \rho_o\omega^2}\left\{\delta_{mn}\ k_1^2\ \frac{\exp(ik_1r)}{r} - \frac{\partial}{\partial x_m}\frac{\partial}{\partial x_n}\left(\frac{\exp(ik_or)}{r} - \frac{\exp(ik_1r)}{r}\right)\right\}$$

$$\overset{\perp}{\sum} = \lambda_o\ \overleftrightarrow{I}\nabla \cdot \overleftrightarrow{G} + \mu_o(\nabla\overleftrightarrow{G} + \overleftrightarrow{G}\nabla)$$

or

$$\sum\nolimits_{\ell mn} = \lambda_o\ \delta_{\ell m}\frac{\partial}{\partial x_k} G_{kn} + \mu_o\left(\frac{\partial}{\partial x_\ell} G_{mn} + \frac{\partial}{\partial x_m} G_{\ell n}\right)$$

Equation (67) constitutes 3 conditions sufficient to determine the unknowns "b", "g", and "f" (or rather elimination of "g" and "f" to allow determination of "b"). The integral must be evaluated for n x m surface points to yield a matrix containing 3 x m x n elements.

The advantage of this technique is the reduction in the number of unknowns. The disadvantage is the evaluation of the integral. Unfortunately, no use of orthgonality can be employed to reduce the numerical computation of the matrix elements. Greater accuracy can in theory be realized however, due to the larger allowable m x n product.

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